Introduction to QCD at Colliders Lecture IV: The production of W, Z and heavy quarks at colliders

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Slides available from http://theory.fnal.gov/people/ellis/Talks/Fermi06

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The production of W, Z and Heavy quarks

- Multiplicity growth with angular ordering
- Hadron-hadron processes and factorization
- W production
  - ⋆ DY cross section
  - ⋆ Subtraction method
- W + jet production
- Heavy quark decays
- Heavy quark production

Multiplicity growth

- Angular ordering condition, on pseudo-offshellness, $\tilde{t_b} < z^2\tilde{t}$, $\tilde{t} = E^2(1-\cos\theta)$.

$$D_i(x,t) = D_i(x,t_0) + \sum_j \int_x^1 \frac{dz}{z} \int_{t_0}^{z^2 t} \frac{dt'}{t'} \frac{\alpha_S}{2\pi} P_{ji}(z,\alpha_S) D_j(x/z,t')\ ,$$

or in differential form

$$t\frac{\partial}{\partial t} D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} P_{ji}(z,\alpha_S) D_j(x/z,z^2 t)\ .$$

Notice that this differs from the conventional evolution equation only in the z-dependent change of scale on the right-hand side. Consider first the solution of taking α_S fixed and neglecting the sum over different branchings. Then we have

$$t\frac{\partial}{\partial t}\tilde{D}(j,t) = \frac{\alpha_S}{2\pi} \int_x^1 dz\, z^{j-1} P(z) \tilde{D}(j,z^2 t)\ .$$

- we try a solution of the form $D(j,t) \propto t^{\gamma(j,\alpha_S)}$

Resummed anomalous dimension

- we find that the anomalous dimension $\gamma(j, \alpha_S)$ must satisfy the implicit equation

$$\gamma(j, \alpha_S) = \frac{\alpha_S}{2\pi} \int_0^1 dz\, z^{j-1+2\gamma(j,\alpha_S)} P(z)\ .$$

$$\gamma_{gg}(j, \alpha_S) = \frac{C_A \alpha_S}{\pi} \frac{1}{j - 1 + 2\gamma_{gg}(j, \alpha_S)}$$

$$\begin{aligned} \gamma_{gg}(j, \alpha_S) \; &= \; \frac{1}{4} \left[\sqrt{(j-1)^2 + \frac{8 C_A \alpha_S}{\pi}} - (j-1) \right] \\ &= \sqrt{\frac{C_A \alpha_S}{2\pi}} - \frac{1}{4}(j-1) + \dots\ . \end{aligned}$$

- at any fixed $j \neq 1$ we can expand in a different way for sufficiently small α_S

$$\gamma_{gg}(j, \alpha_S) = \frac{C_A \alpha_S}{\pi} \frac{1}{(j-1)} - 2 \left(\frac{C_A \alpha_S}{\pi} \right)^2 \frac{1}{(j-1)^3} + \dots\ .$$

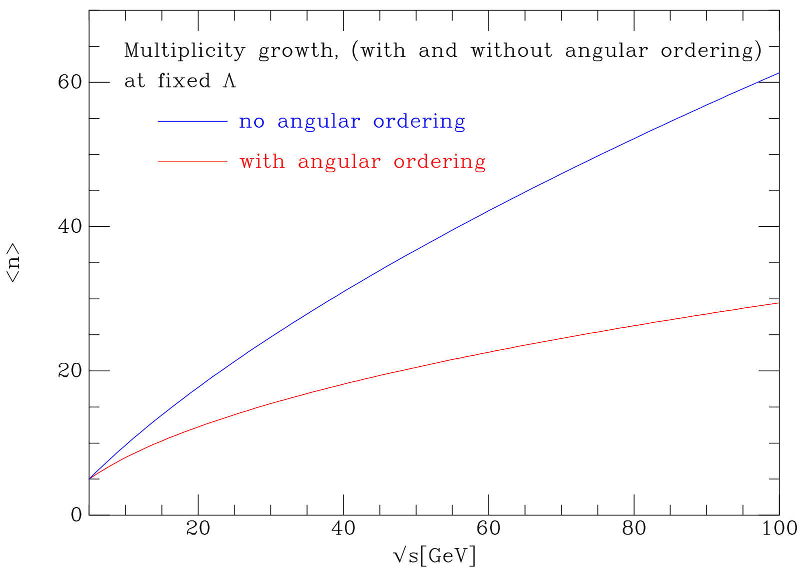
This series displays the terms that are most singular as $j \to 1$ in each order.

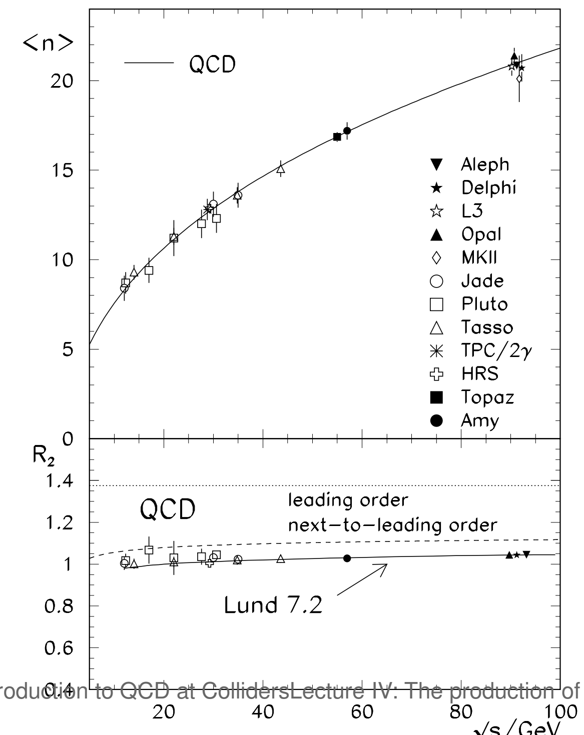
Multiplicity growth II

- we can introduce running α by writing
  $\tilde{D}(j,t) \sim t^{\gamma_{gg}(j,\alpha_S)} = \exp\left[\int^t \gamma_{gg}(j,\alpha_S)\frac{dt'}{t'}\right].$
- we replace $\gamma_{gg}(j,\alpha_S)$ in the integrand by $\gamma_{gg}(j,\alpha_S(t'))$.

$$\int^t \gamma_{gg}(j,\alpha_S(t'))\frac{dt'}{t'} = \int^{\alpha_S(t)} \frac{\gamma_{gg}(j,\alpha_S)}{\beta(\alpha_S)}\,d\alpha_S, \quad \beta(\alpha_S) = -b\alpha_S^2 + \ldots,$$

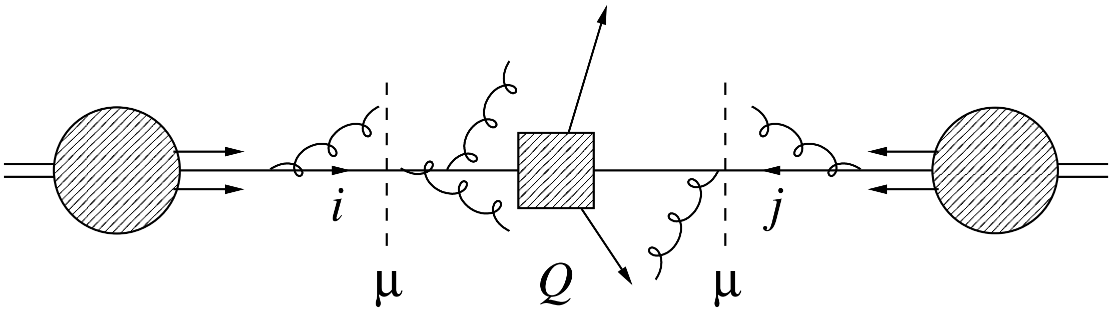
$$\langle n(s)\rangle \sim \exp\left[\frac{1}{b}\sqrt{\frac{2C_A}{\pi\alpha_S(s)}}\right] \sim \exp\sqrt{\frac{6}{\pi b}\ln\frac{s}{\Lambda^2}}$$





Hadron-hadron processes

- In hard hadron-hadron scattering, constituent partons from each incoming hadron interact at short distance (large momentum transfer Q^2).



- For hadron momenta P_1, P_2 ($S = 2P_1 \cdot P_2$), form of cross section is

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 D_i(x_1, \mu^2) D_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_S(\mu^2), Q^2/\mu^2)$$

where μ^2 is factorization scale and $\hat{\sigma}_{ij}$ is subprocess cross section for parton types i, j.

- ⋆ Notice that factorization scale is in principle arbitrary: affects only what we call part of subprocess or part of initial-state evolution (parton shower).
- ⋆ Unlike e^+e^- or ep, we may have interaction between spectator partons, leading to soft underlying event and/or multiple hard scattering.

Factorization of the cross section

- Why does the factorization property hold and when it should fail?
- For a heuristic argument, consider the simplest hard process involving two hadrons

$$H_1(P_1) + H_2(P_2) \to V + X.$$

- Do the partons in hadron H_1, through the influence of their colour fields, change the distribution of partons in hadron H_2 before the vector boson is produced? Soft gluons which are emitted long before the collision are potentially troublesome.
- A simple model from classical electrodynamics. The vector potential due to an electromagnetic current density J is given by

$$A^\mu(t,\vec{x}) = \int dt' d\vec{x}' \; \frac{J^\mu(t',\vec{x}')}{|\vec{x}-\vec{x}'|} \; \delta(t' + |\vec{x}-\vec{x}'| - t) \; ,$$

where the delta function provides the retarded behaviour required by causality.

- Consider a particle with charge e travelling in the positive z direction with constant velocity β. The non-zero components of the current density are

$$
\begin{aligned}
J^t(t',\vec{x}') &= e\delta(\vec{x}' - \vec{r}(t'))\ , \\
J^z(t',\vec{x}') &= e\beta\delta(\vec{x}' - \vec{r}(t')),\ \ \vec{r}(t') = \beta t'\hat{z},
\end{aligned}
$$

$\hat{z}$ is a unit vector in the z direction. At an observation point (the supposed position of hadron H_2) described by coordinates x, y and z, the vector potential (either performing the integrations using the current density given above, or by Lorentz transformation of the scalar potential in the rest frame of the particle) is

$$
\begin{aligned}
A^t(t,\vec{x}) &= \frac{e\gamma}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}} \\
A^x(t,\vec{x}) &= 0 \\
A^y(t,\vec{x}) &= 0 \\
A^z(t,\vec{x}) &= \frac{e\gamma\beta}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}},
\end{aligned}
$$

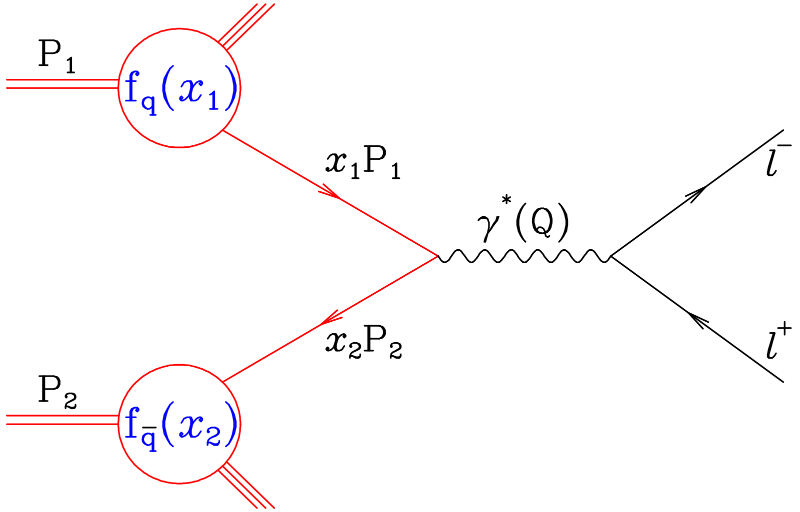
where $\gamma^2 = 1/(1-\beta^2)$. Target hadron H_2 is at rest near the origin, so that $\gamma \approx s/m^2$.

- Note that for large γ and fixed non-zero $(\beta t - z)$ some components of the potential tend to a constant independent of γ, suggesting that there will be non-zero fields which are not in coincidence with the arrival of the particle, even at high energy.
- However at large γ the potential is a pure gauge piece, $A^\mu = \partial^\mu \chi$ where χ is a scalar function
- Covariant formulation using the vector potential A has large fields which have no effect.
- For example, the electric field along the z direction is

$$E^z(t, \vec{x}) = F^{tz} \equiv \frac{\partial A^z}{\partial t} + \frac{\partial A^t}{\partial z} = \frac{e\gamma(\beta t - z)}{[x^2 + y^2 + \gamma^2(\beta t - z)^2]^{\frac{3}{2}}}.$$

The leading terms in γ cancel and the field strengths are of order $1/\gamma^2$ and hence of order m^4/s^2. The model suggests the force experienced by a charge in the hadron H_2, at any fixed time before the arrival of the quark, decreases as m^4/s^2.

Lepton-pair production



- Mechanism for Lepton pair production, W-production, Z-production, Vector-boson pairs, ...
- Collectively known as the Drell-Yan process.
- Colour average $1/N$.

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{\sigma_0}{N} Q_q^2 \, \delta(\hat{s} - Q^2), \qquad \sigma_0 = \frac{4\pi\alpha^2}{3Q^2}, \text{ cf } e^+e^- \text{ annihilation.}$$

In the CM frame of the two hadrons, the momenta of the incoming partons are

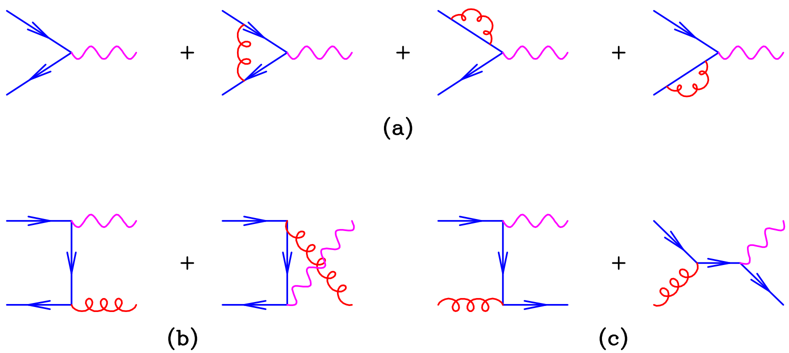
$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1), \quad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2) \, .$$

The square of the $q\bar{q}$ collision energy $\hat{s}$ is related to the overall hadron-hadron collision energy by $\hat{s} = (p_1 + p_2)^2 = x_1 x_2 s$. The parton-model cross section for this process is:

$$\begin{aligned}
\frac{d\sigma}{dM^2} &= \int_0^1 dx_1 dx_2 \sum_q \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\} \frac{d\hat{\sigma}}{dM^2}(q\bar{q} \to l^+ l^-) \\
&= \frac{\sigma_0}{Ns} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \, \delta(1-z) \left[\sum_q Q_q^2 \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\} \right] .
\end{aligned}$$

- For later convenience we have introduced the variable $z = \frac{Q^2}{\hat{s}} = \frac{Q^2}{x_1 x_2 s}$.
- The sum here is over quarks only and the $\bar{q}q$ contributions are indicated explicitly.

Next-to-leading order



- The contribution of the real diagrams (in four dimensions) is

$$|M|^2 \sim g^2 C_F \left[\frac{u}{t} + \frac{t}{u} + \frac{2Q^2 s}{ut} \right] = g^2 C_F \left[\left(\frac{1+z^2}{1-z} \right) \left(\frac{-s}{t} + \frac{-s}{u} \right) - 2 \right]$$

  where $z = Q^2/s$, $s + t + u = Q^2$.

- Note that the real diagrams contain collinear singularities, $u \to 0, t \to 0$ and soft singularities, $z \to 1$.
- The coefficient of the divergence is the unregulated branching probability $\hat{P}_{qq}(z)$.
- Ignore for simplicity the diagrams with incoming gluons.

- Control the divergences by continuing the dimensionality of space-time, $d = 4 - 2\epsilon$, (technically this is dimensional reduction). Performing the phase space integration, the total contribution of the real diagrams is

$$\begin{aligned}\sigma_R &= \frac{\alpha_S}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^\epsilon c_\Gamma \Bigg[\Big(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{\pi^2}{3}\Big)\delta(1-z) - \frac{2}{\epsilon} P_{qq}(z) \\ &- 2(1-z) + 4(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - 2\frac{1+z^2}{(1-z)}\ln z\Bigg]\end{aligned}$$

  with $c_\Gamma = (4\pi)^\epsilon / \Gamma(1-\epsilon)$.

- The contribution of the virtual diagrams is

$$\sigma_V = \delta(1-z)\left[1 + \frac{\alpha_S}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^\epsilon c'_\Gamma \Big(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 6 + \pi^2\Big)\right]$$

  $c'_\Gamma = c_\Gamma + O(\epsilon^3)$

- Adding it up we get in dim-reduction

$$\sigma_{R+V} = \frac{\alpha_S}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} c_\Gamma \Bigg[\Big(\frac{2\pi^2}{3} - 6\Big)\delta(1-z) - \frac{2}{\epsilon} P_{qq}(z) - 2(1-z)$$
$$+ \quad 4(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - 2\frac{1+z^2}{(1-z)} \ln z \Bigg]$$

- The divergences, proportional to the branching probability , are universal.
- We will factorize them into the parton distributions. We perform the mass factorization by subtracting the counterterm

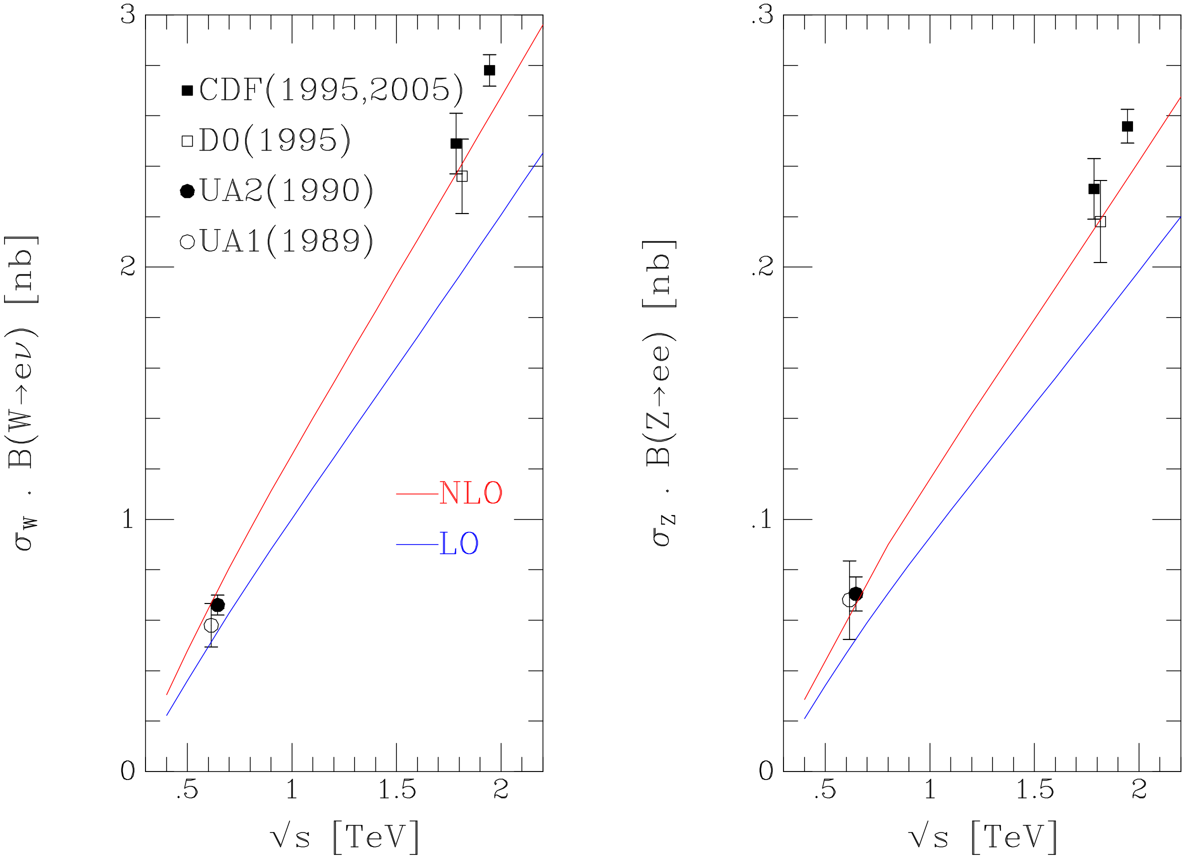
$$2\frac{\alpha_S}{2\pi} C_F \left[\frac{-c_\Gamma}{\epsilon} P_{qq}(z) - (1-z) + \delta(1-z) \right]$$

(The finite terms are necessary to get us to the $\overline{MS}$-scheme).

$$\hat{\sigma} = \frac{\alpha_S}{2\pi} C_F \left[\Big(\frac{2\pi^2}{3} - 8\Big)\delta(1-z) + 4(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - 2\frac{1+z^2}{(1-z)} \ln z + 2 P_{qq}(z) \ln \frac{Q^2}{\mu^2} \right]$$

- Similar correction for incoming gluons.

Application to W, Z production



- Agreement with NLO theory is good.
- LO curves lie about 25% too low.
- NNLO results are also known and lead to a further modest (4%) increase at the Tevatron.

General calculational method for NLO

- Direct integration is good for the total cross section, but for differential distributions, (to which we want to apply cuts), we need a Monte Carlo method.
- We use a general subtraction procedure at NLO.
- at NLO the cross section for two initial partons a and b and for m outgoing partons, is given by

$$\sigma_{ab} = \sigma_{ab}^{LO} + \sigma_{ab}^{NLO}$$

where

$$\begin{aligned}
\sigma_{ab}^{LO} &= \int_m d\sigma_{ab}^{B} \\
\sigma_{ab}^{NLO} &= \int_{m+1} d\sigma_{ab}^{R} + \int_m d\sigma_{ab}^{V}
\end{aligned}$$

the singular parts of the QCD matrix elements for real emission, corresponding to soft and collinear emission can be isolated in a process independent manner

Calculational method (cont)

- One can use this the construct a set of counterterms

$$d\sigma^{ct} = \sum_{ct} \int_m d\sigma^B \otimes \int_1 dV_{ct}$$

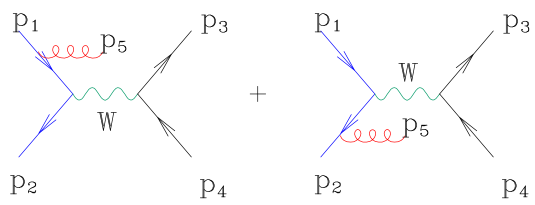
  where $d\sigma^B$ denotes the appropriate colour and spin projection of the Born-level cross section, and the counter-terms are independent of the details of the process under consideration.

- these counterterm cancel all non-integrable singularities in $d\sigma^R$, so that one can write

$$\sigma_{ab}^{NLO} = \int_{m+1} [d\sigma_{ab}^R - d\sigma_{ab}^{ct}] + \int_{m+1} d\sigma_{ab}^{ct} + \int_m d\sigma_{ab}^V$$

  The phase space integration in the first term can be performed numerically in four dimensions.

Matrix element counter-event for W production



In the soft limit $p_5 \to 0$ we have

$$|M_1(p_1,p_2,p_3,p_4,p_5)|^2 = g^2 C_F \frac{p_1 \cdot p_2}{p_1 \cdot p_5 \; p_2 \cdot p_5} |M_0(p_1,p_2,p_3,p_4)|^2$$

- Eikonal factor can be associated with radiation from a given leg by partial fractioning

$$\frac{p_1 \cdot p_2}{p_1 \cdot p_5 \; p_2 \cdot p_5} = [\frac{p_1 \cdot p_2}{p_1 \cdot p_5 + p_2 \cdot p_5}][\frac{1}{p_1 \cdot p_5} + \frac{1}{p_2 \cdot p_5}]$$

- including the collinear contributions, singular as $p_1 \cdot p_5 \to 0$, the matrix element for the counter event has the structure

$$|M_1(p_1,p_2,p_3,p_4,p_5)|^2 = \frac{g^2}{x_a p_1 \cdot p_5} \hat{P}_{qq}(x_a) |M_0(x_a p_1, p_2, \tilde{p}_3, \tilde{p}_4)|^2$$

where $1 - x_a = (p_1 \cdot p_5 + p_2 \cdot p_5)/p_1 \cdot p_2$ and $\hat{P}_{qq}(x_a) = C_F(1+x^2)/(1-x)$

Subtraction method for NLO

- For event $q(p_1) + \bar{q}(p_2) \to W^+(\nu(p_3) + e^+(p_4)) + g(p_5)$ with $p_1 + p_2 = \sum_{i=3}^5 p_i$
- generate a counter event $q(x_a p_1) + \bar{q}(p_2) \to W^+(\nu(\tilde{p}_3) + e^+(\tilde{p}_4))$ and $x_a p_1 + p_2 = \sum_{i=3}^4 \tilde{p}_i$ with $1 - x_a = (p_1 \cdot p_5 + p_2 \cdot p_5)/p_1 \cdot p_2$.
- A Lorentz transformation is performed on all j final state momenta $\tilde{p}_j = \Lambda^\mu_\nu p^\nu_j, j = 3, 4$ such that $\tilde{p}^\mu_j \to p^\mu_j$ for p_5 collinear or soft.
- The longitudinal momentum of p_5 is absorbed by rescaling with x.
- The other components of the momentum, p_5 are absorbed by the Lorentz transformation.
- In terms of these variables the phase space has a convolution structure,

$$d\phi^{(3)}(p_1, p_2; p_3, p_4, p_5) = \int_0^1 dx\; d\phi^{(2)}(p_2, xp_1; \tilde{p_3}, \tilde{p_4})[dp_5(p_1, p_2, x)]$$

where

$$[dp_5(p_1, p_2, x_a)] = \frac{d^d p_5}{(2\pi)^3} \delta^+(p_5^2)\Theta(x)\Theta(1-x)\delta(x - x_a)$$

- If k_i is the emitted parton, and p_a, p_b are the incoming momenta, define the shifted momenta

$$\widetilde{k}_j^\mu = k_j^\mu - \frac{2k_j \cdot (K + \widetilde{K})}{(K + \widetilde{K})^2} \, (K + \widetilde{K})^\mu + \frac{2k_j \cdot K}{K^2} \, \widetilde{K}^\mu \ ,$$

where the momenta K^μ and $\widetilde{K}^\mu$ are,

$$K^\mu = p_a^\mu + p_b^\mu - p_i^\mu \ \ , \widetilde{K}^\mu = \widetilde{p}_{ai}^\mu + p_b^\mu \ \ .$$

- Since $2\sum_j k_j \cdot K = 2K^2$ and $2\sum_j k_j \cdot (K + \widetilde{K}) = 2K^2 + 2K \cdot \widetilde{K} = (K + \widetilde{K})^2$ the momentum conservation constraint in the $m+1$-parton matrix

$$p_a^\mu + p_b^\mu - \sum_j k_j^\mu - p_i^\mu = 0 \ \ .$$

implies

$$\widetilde{p}_{ai}^\mu + p_b^\mu - \sum_j \widetilde{k}_j^\mu = 0 \ \ .$$

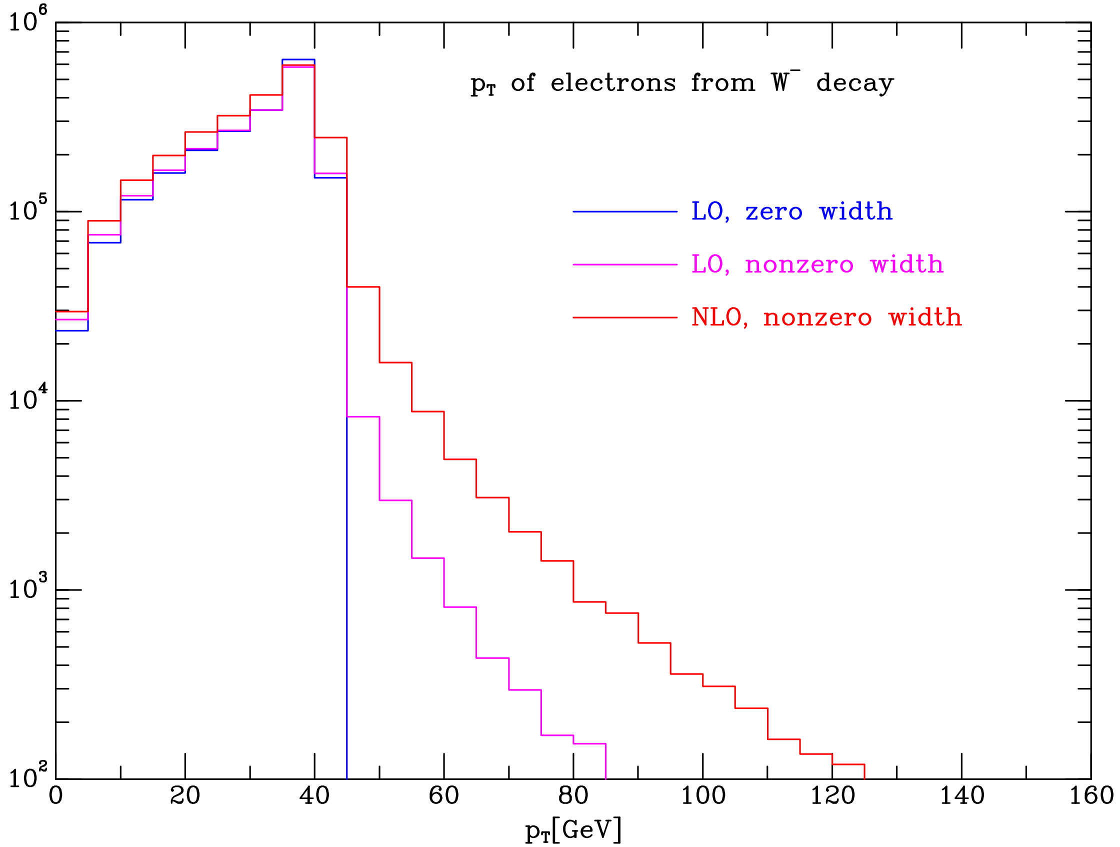
- Note also that the shifted momenta can be rewritten in the following way:

$$\widetilde{k}_j^\mu = \Lambda^\mu{}_\nu(K,\widetilde{K})\, k_j^\nu \quad ,$$

$$\Lambda^\mu{}_\nu(K,\widetilde{K}) = g^\mu{}_\nu - \frac{2(K+\widetilde{K})^\mu (K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2\widetilde{K}^\mu K_\nu}{K^2} \quad ,$$

- the matrix $\Lambda^\mu{}_\nu(K,\widetilde{K})$ generates a proper Lorentz transformation on the final-state momenta.
- If the emitted parton has zero transverse momenta, the Lorentz transformation reduces to the identity.

Why NLO?



- Calculation of NLO corrections, give a better prediction for the rate.
- Extra radiation can modify kinematic distributions.

MCFM overview

John Campbell and R.K. Ellis

- Parton level cross-sections predicted to NLO in α_S

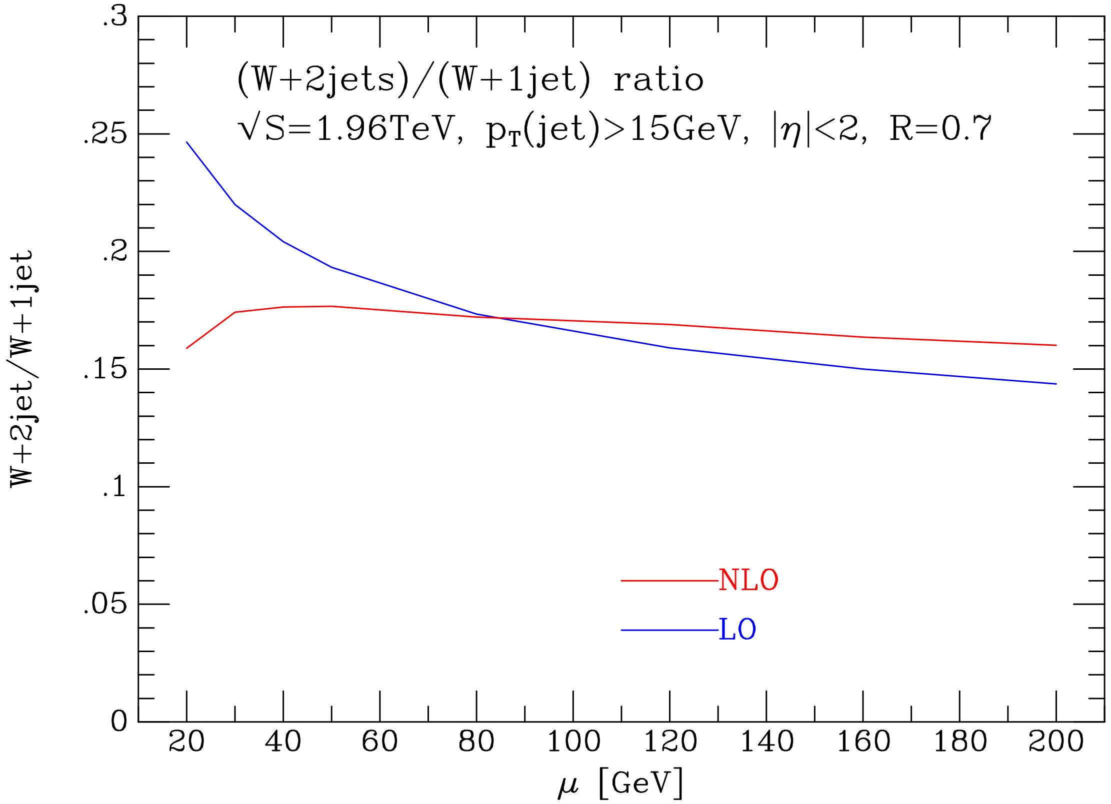
$p\bar{p} \to W^\pm/Z$	$p\bar{p} \to W^+ + W^-$
$p\bar{p} \to W^\pm + Z$	$p\bar{p} \to Z + Z$
$p\bar{p} \to W^\pm + \gamma$	$p\bar{p} \to W^\pm/Z + H$
$p\bar{p} \to W^\pm + g^\star\ (\to b\bar{b})$	$p\bar{p} \to Zb\bar{b}$
$p\bar{p} \to W^\pm/Z +$ 1 jet	$p\bar{p} \to W^\pm/Z +$ 2 jets
$p\bar{p}(gg) \to H$	$p\bar{p}(gg) \to H +$ 1 jet
$p\bar{p}(VV) \to H +$ 2 jets	$p\bar{p} \to t + X$
$pp \to t + W$	

$\oplus$ less sensitivity to μ_R, μ_F, rates are better normalized, fully differential distributions.

$\ominus$ low particle multiplicity (no showering), no hadronization, hard to model detector effects

MCFM:examples

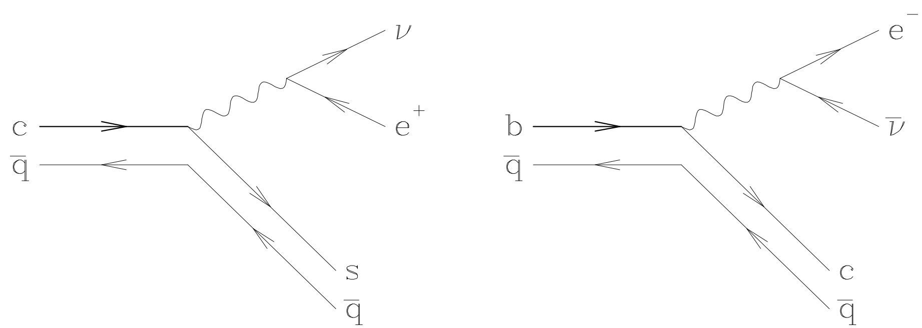
- (W+2 jet)/(W+1 jet)



Charm and bottom quark decays

- treat the semi-leptonic decays of hadrons containing c and b quarks in analogy with the decay of a free muon, (*spectator model*)
- Lagrangians for CKM-favoured decays are

$$
\begin{aligned}
\mathcal{L}^{(c)} &= -\frac{G_F}{\sqrt{2}} V_{cs}\ \bar{s}\gamma^\mu(1-\gamma_5)c\ \bar{\nu}\gamma_\mu(1-\gamma_5)e\ , \\
\mathcal{L}^{(b)} &= -\frac{G_F}{\sqrt{2}} V_{cb}\ \bar{c}\gamma^\mu(1-\gamma_5)b\ \bar{e}\gamma_\mu(1-\gamma_5)\nu\ .
\end{aligned}
$$



$$
\begin{aligned}
\overline{\sum}|\mathcal{M}^{(c)}|^2 &= 64\ G_F^2\ |V_{cs}|^2\ c\cdot e^+\ s\cdot\nu\ , \\
\overline{\sum}|\mathcal{M}^{(b)}|^2 &= 64\ G_F^2\ |V_{cb}|^2\ b\cdot\bar{\nu}\ c\cdot e^-\ ,
\end{aligned}
$$

where $b, c, s, \nu, \bar{\nu}, e^+$ and e^- now stand for the four-momenta of the particles in the decay.

- by angular momentum conservation s and ν momenta prefer to be anti-parallel in c quark decay. The endpoint configuration in which the e^+ recoils against the parallel s and ν is thus disfavoured. We expect a soft spectrum for the positron.
- Conversely we expect a hard spectrum for the neutrino (or for the electron coming from the decay of a b quark).

$$\Gamma_{\text{sl}}^{(Q)} = \frac{m_Q}{2^8\pi^3} \int dxdy\, \theta(x+y-x_m)\, \theta(x_m-x-y+xy)\, \overline{\sum}|\mathcal{M}^{(Q)}|^2$$

- x and y are the rescaled energies of the charged and neutral leptons, $x = 2E_e/m_Q$, $y = 2E_\nu/m_Q$ in the frame in which the heavy quark Q is at rest. The kinematic endpoint of the spectrum is denoted by x_m and is given by $x_m = 1-\epsilon^2$ where $\epsilon = m_q/m_Q$. The result for the semi-leptonic widths of the c and the b is

$$\begin{aligned} \frac{d\Gamma_{\text{sl}}^{(c)}}{dxdy} &= |V_{cs}|^2\Gamma_0(m_c)[12x(x_m-x)] \\ \frac{d\Gamma_{\text{sl}}^{(b)}}{dxdy} &= |V_{cb}|^2\Gamma_0(m_b)[12y(x_m-y)] \end{aligned}$$

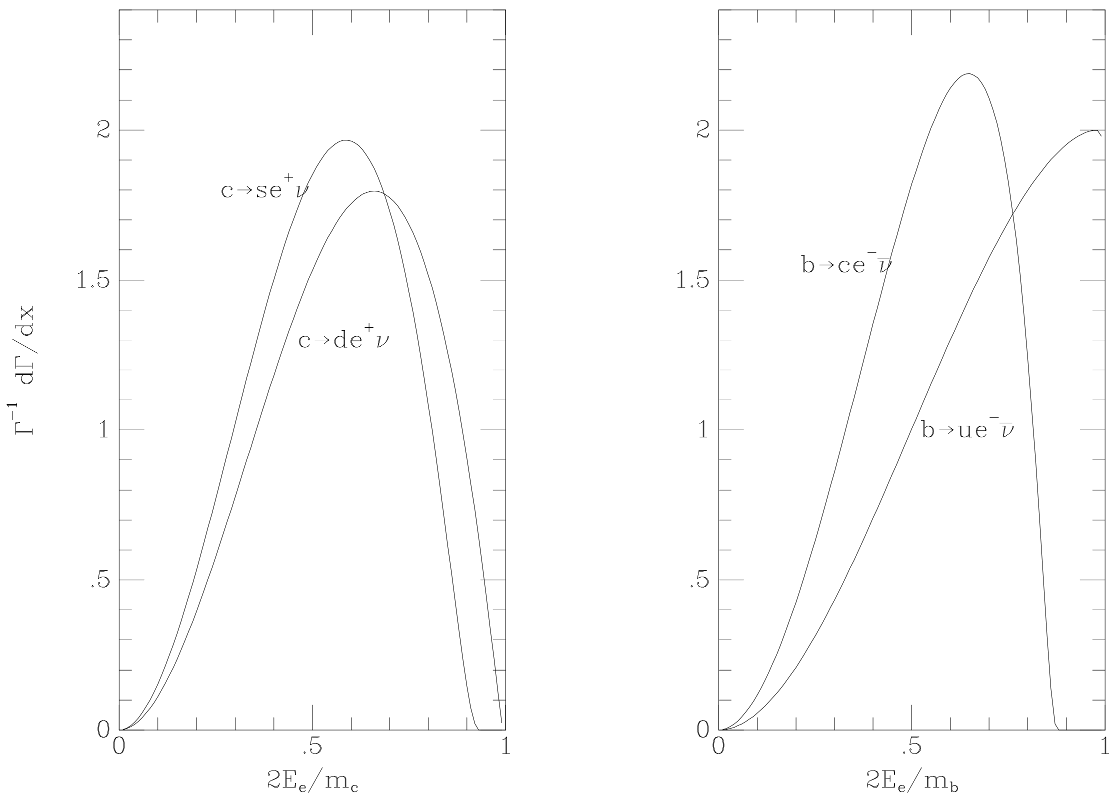
Γ_0 is the rescaled muon decay rate,

$$\Gamma_0(m_Q) = \frac{G_F^2 m_Q^5}{192\pi^3} \ .$$

Integrating over y the charged lepton spectra are

$$\frac{d\Gamma_{\mathsf{sl}}^{(c)}}{dx} = |V_{cs}|^2 \Gamma_0(m_c) \left[\frac{12x^2(x_m - x)^2}{(1-x)} \right]$$

$$\frac{d\Gamma_{\mathsf{sl}}^{(b)}}{dx} = |V_{cb}|^2 \Gamma_0(m_b) \left[\frac{2x^2(x_m - x)^2}{(1-x)^3} \right] (6 - 6x + x x_m + 2x^2 - 3x_m).$$



The e^+ from charm decay has a soft spectrum. The e^- from the CKM-disfavoured mode $b \to u$ has a hard spectrum.

- The measurement of leptons with energies beyond the kinematic limit for $b \to c$ gives information about V_{ub}.
- Allowing for theoretical uncertainty in endpoint region, the measured value is $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$

- After integration the semi-leptonic width including mass effects is found to be

$$\Gamma^{(Q)}_{\text{sl}} = |V_{Qq}|^2 \Gamma_0(m_Q) f\left(\frac{m_q}{m_Q}\right)$$

where the function f is given by

$$f(\epsilon) = (1-\epsilon^4)(1-8\epsilon^2+\epsilon^4) - 24\epsilon^4 \ln\epsilon \ .$$

Including the CKM-disfavoured mode $c \to d$ the result for the semi-leptonic decay of the c quark is

$$\Gamma^{(c)}_{\text{sl}} = \Gamma_0(m_c)\Big[f(m_s/m_c)|V_{cs}|^2 + f(m_d/m_c)|V_{cd}|^2\Big].$$

- For a rough estimate ignore strong interaction corrections and choose $m_c = 1.4\ \mathrm{GeV}$. The theoretical estimate for the semi-leptonic width is

$$\Gamma_{\text{sl}} = 1.1 \times 10^{-10}\ \mathrm{MeV}.$$

From the measured semi-leptonic branching ratios of the $D^+, (17.2 \pm 1.9\%)$ and $D^0, (7.7 \pm 1.2\%)$ and the inverse of known lifetimes, we can calculate the semi-leptonic widths.

$$\begin{aligned}\Gamma_{\text{sl}}(D^0) &= (1.22 \pm 0.20) \times 10^{-10}\ \mathrm{MeV} \\ \Gamma_{\text{sl}}(D^+) &= (1.07 \pm 0.13) \times 10^{-10}\ \mathrm{MeV}.\end{aligned}$$

- The spectator model gives a fair description of the semi-leptonic decays of D mesons.

- For the semi-leptonic decays of B mesons, the theoretical decay width is

$$\Gamma_{\textsf{sl}}^{(b)} = \Gamma_0(m_b) f\left(\frac{m_c}{m_b}\right) |V_{cb}|^2 \eta_0,$$

- The CKM-disfavoured mode makes a negligible contribution to the total rate. $\eta_0 \approx 0.87$ due to strong interaction corrections. Using the measured semi-leptonic branching ratios of the $B^{\pm}, (10.1 \pm 1.8 \pm 1.5\%)$ and $B^0, (10.9 \pm 0.7 \pm 1.1\%)$, the semi-leptonic widths of the B mesons are

$$\begin{aligned} \Gamma_{\textsf{sl}}(B^0) &= (0.48 \pm 0.12) \times 10^{-10} \text{ MeV} \\ \Gamma_{\textsf{sl}}(B^+) &= (0.43 \pm 0.18) \times 10^{-10} \text{ MeV}. \end{aligned}$$

We can estimate V_{cb}. Choosing the values $m_b = 4.8 \text{ GeV}, m_c = 1.4 \text{ GeV}$ and $V_{cb} = 0.04$ one obtains

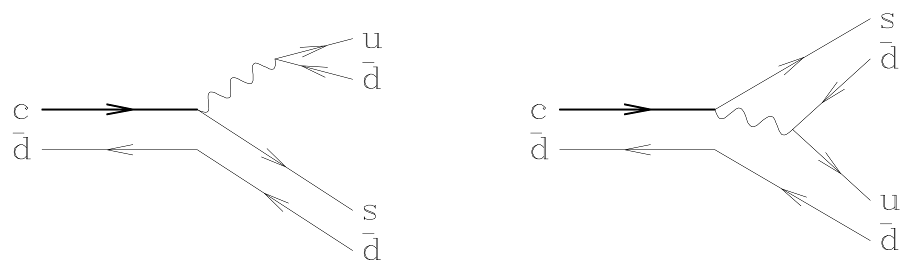
$$\Gamma_{\textsf{sl}} = 2.7 \times 10^{-8} |V_{cb}|^2 \text{ MeV} = 0.44 \times 10^{-10} \text{ MeV} .$$

Hadronic decays

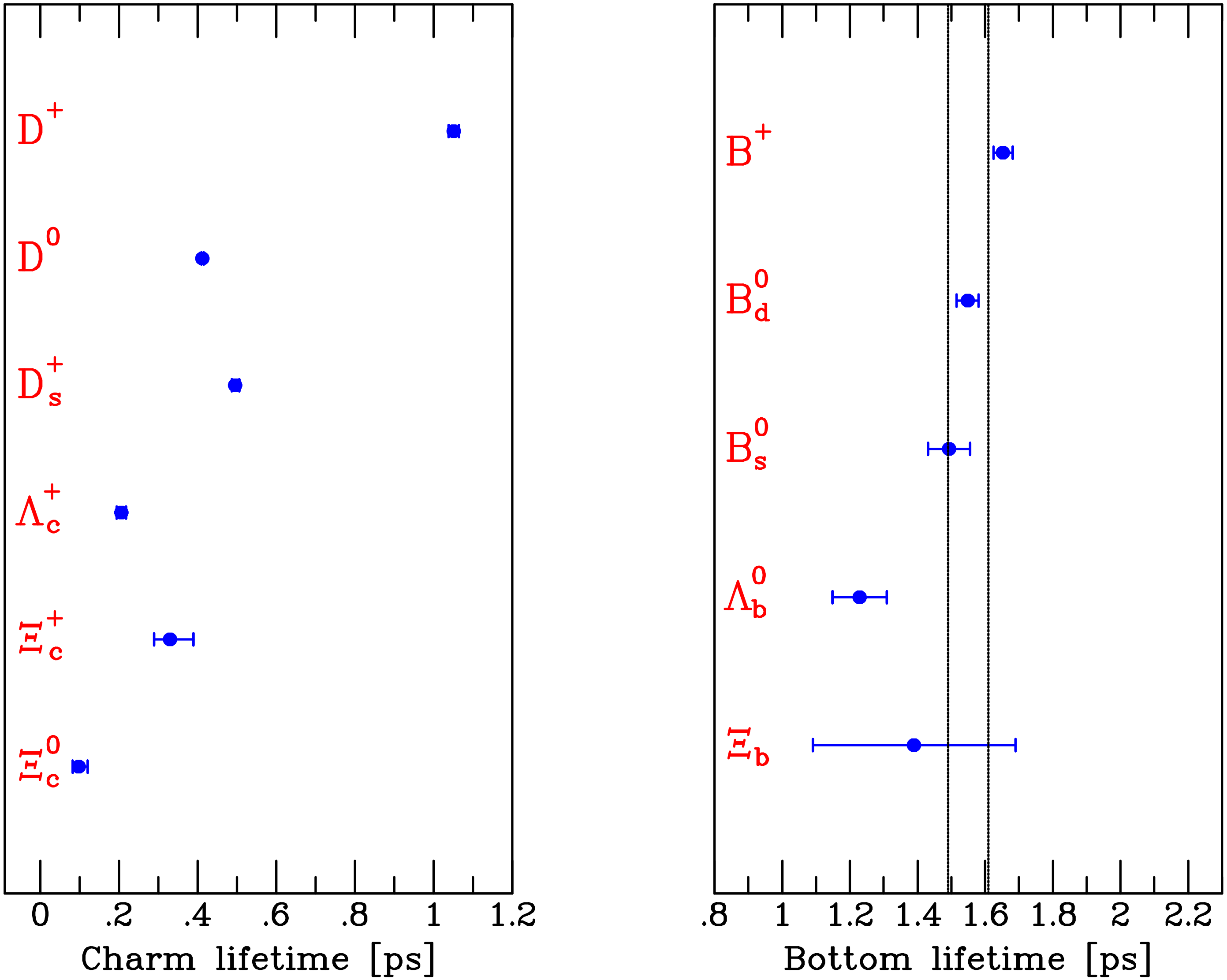
- Estimate for the total width using the spectator model. The width in the spectator model is given simply by the weak decay of the heavy quark followed by the subsequent decay of the resulting virtual W boson.
- Diagrams involving spectators are suppressed by powers of the heavy quark mass. For example, in the decay of a D_s^+ meson, a non-spectator diagram would result from the annihilation of the charm quark with the anti-strange quark.

$$BR(c \to eX) = \frac{1}{1+1+3} ,$$

ignoring strong interaction effects. Therefore the prediction of the simplest spectator model for the total width of a charmed hadron is given by multiplying the semi-leptonic width by five.
- This leads to an expected common lifetime for all charmed hadrons of the order of 1.2 ps. However the lifetimes of the D^+ and the D^0 mesons are very different.



Measured lifetimes



- Since the semi-leptonic widths are approximately equal, we can conclude that the failure of the spectator model is due to differences in the hadronic widths of the charmed hadrons.
- Reasons for the failure of the spectator model for D decays include non-spectator diagrams and strong radiative corrections.
- Interference effects should be suppressed by powers of m_c because of the small overlap of the $\bar{d}$ coming from the charm decay with the spectator $\bar{d}$. The former is initially localized in a volume of order $1/m_c^3$, whereas the latter is distributed throughout the D meson state.
- Since these interference effects appear to be important, we conclude that the charm quark is too light to be treated as a heavy quark in this context.

B total width

- Apply the spectator model to hadronic B decays.
- One might expect the B lifetime to be a factor of $(m_c/m_b)^5$ shorter than the estimate for the charm quark lifetime given above. However this mass effect is almost entirely cancelled by the factor of $|V_{cb}|^2$ which occurs in the expression for the width.
- The calculation of the semi-leptonic branching ratio for B decays is complicated even in the spectator model, because of the many channels which are kinematically allowed for the decay. A detailed calculation gives $BR(b \to eX) > 12.5\%$.
- The ground-state hadrons containing b quarks have roughly equal lifetimes.
- Calculating the total width from the theoretical semi-leptonic width and the measured semi-leptonic branching ratio, we obtain a lifetime of about 1.6 ps.
- The measured average b lifetime is 1.55 ± 0.06 ps
- This corresponds to a proper lifetime expressed in units of length of $c\tau = 463 \pm 18$ μm. A b quark with momentum 20 GeV has a relativistic γ factor of about 4. A B meson of this momentum decaying after one lifetime will travel about 1.9 mm. This decay distance large enough to be measured with a detector, typically a silicon vertex detector.

Top quark decays

- Standard Model. Since $m_t > M_W + m_b$ a top quark decays predominantly into a b quark and an on-shell W boson

$$
\begin{aligned}
t \quad &\rightarrow \quad W^+ + b \\
& \qquad \hookrightarrow l^+ + \nu
\end{aligned}
$$

$$
\begin{aligned}
t \quad &\rightarrow \quad W^+ + b \\
& \qquad \hookrightarrow q + \bar{q}
\end{aligned}
$$

- the branching ratio to leptons is given by counting the decay modes of the W, ($e\bar{\nu}_e$, $\mu\bar{\nu}_\mu$, $\tau\bar{\nu}_\tau$ and three colours of $u\bar{d}$ and $c\bar{s}$,

$$\mathrm{BR}(W^+ \rightarrow e^+\bar{\nu}) = \frac{1}{3+3+3} \approx 11\%.$$

With a perfect detector the numbers of events expected at $\sqrt{s} = 1.96$ TeV per fb^{-1} are

$$
\begin{aligned}
\text{N}(e\mu\ \text{+jets}) = &\quad 2 \times .11 \times .11 \times 7500 \approx 180 \\
\text{N}(e\ +\ \text{jets}) = &\quad 2 \times .11 \times .66 \times 7500 \approx 1100.
\end{aligned}
$$

The existence of both of these decay modes with the correct ratio is a first test of the decay modes of the top.

The W boson coming from top decay can be either left-handed (L) or longitudinally (0) polarized.

$$\overline{\sum}|\mathcal{M}_L|^2 = \frac{2G_F m_t^4}{\sqrt{2}}|V_{tb}|^2\left[2x^2(1-x^2+y^2)\right]$$
$$\overline{\sum}|\mathcal{M}_0|^2 = \frac{2G_F m_t^4}{\sqrt{2}}|V_{tb}|^2\left[1-x^2-y^2(2+x^2-y^2)\right],$$

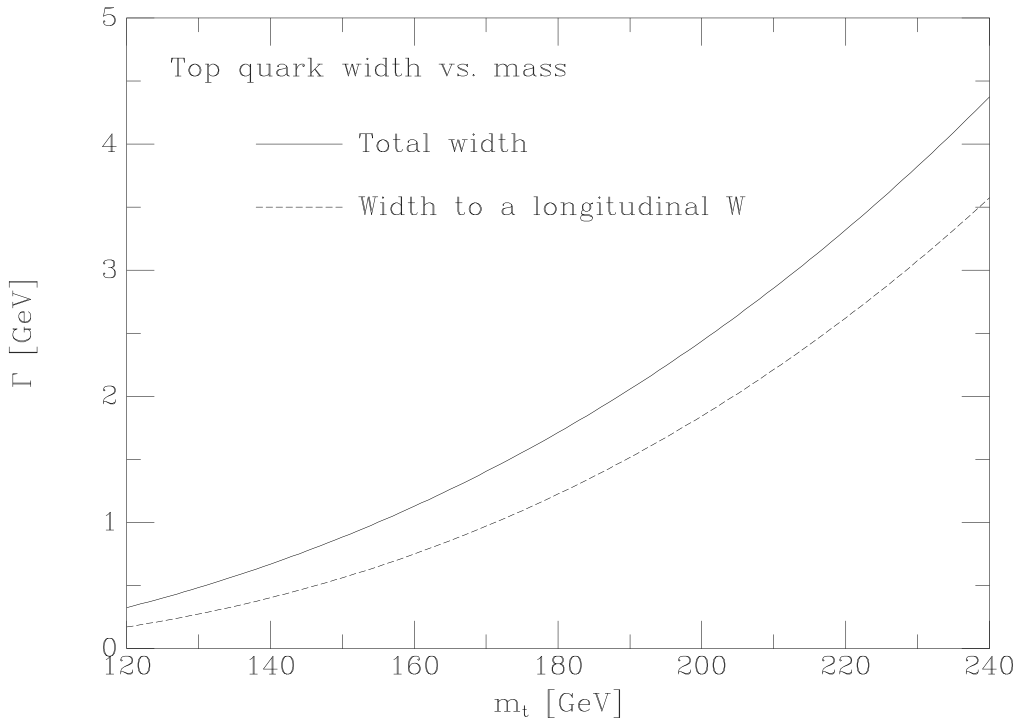
where $x = M_W/m_t, y = m_b/m_t$.
In the limit $m_t \gg M_W$ the result for the total width is

$$\Gamma(t \to bW) = \frac{G_F m_t^3}{8\pi\sqrt{2}}|V_{tb}|^2 \approx 1.76 \text{ GeV}\left(\frac{m_t}{175 \text{ GeV}}\right)^3.$$

$V_{tb} \approx 1$ as suggested by the unitarity relation

$$|V_{tb}|^2 + |V_{cb}|^2 + |V_{ub}|^2 = 1.$$

- The top quark has a ‘semi-weak’ decay rate.
- The lifetime of the top quark is only of order 10^{-25} seconds ($c\tau \sim 10^{-10}\mu$m) and it therefore decays before it has time to hadronize.



- The polarization state of the W controls the angular distribution of the leptons into which it decays. We may define the lepton helicity angle θ_e^*, which is the angle of the charged lepton in the rest frame of the W, with respect to the original direction of travel of the W (*i.e.* anti-parallel to the recoiling b quark). If the b quark jet is identified, this angle can be defined experimentally as

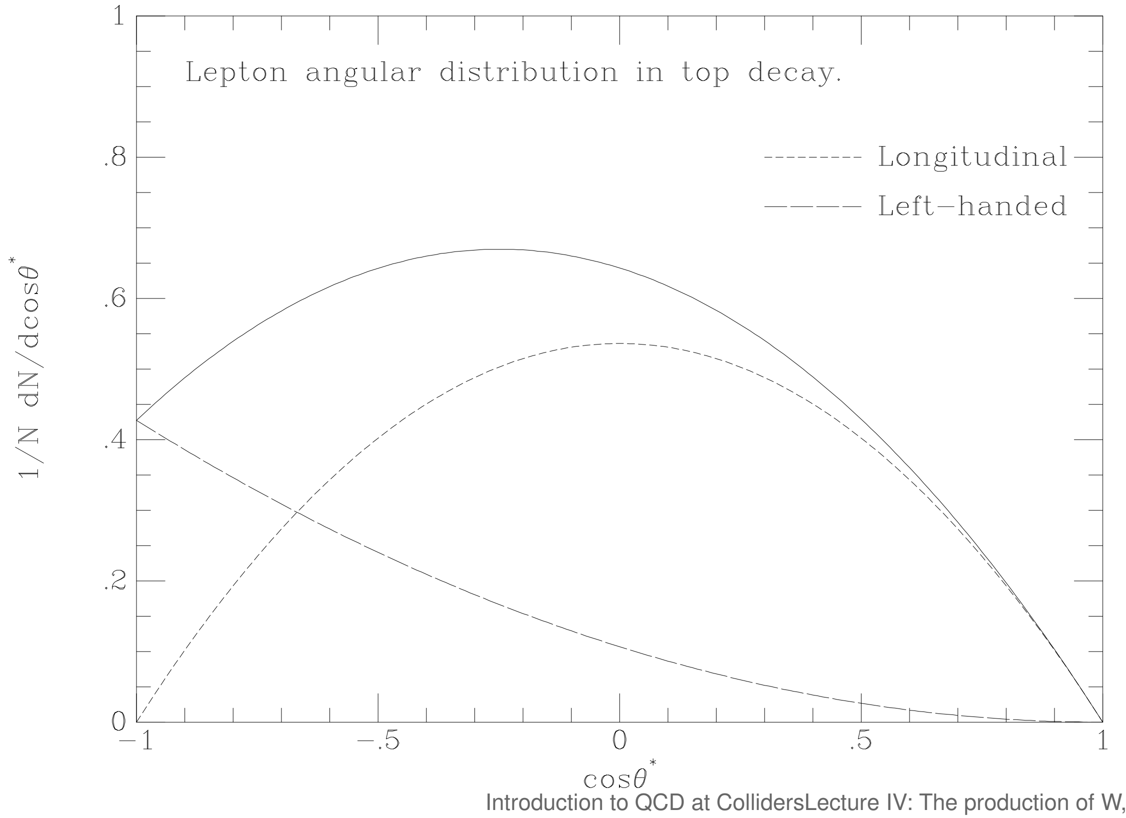
$$\cos\theta_e^* \approx \frac{b\cdot(e^+ - \nu)}{b\cdot(e^+ + \nu)} \approx \frac{4b\cdot e^+}{m_t^2 - M_W^2} - 1,$$

in an obvious notation where t, b, e^+ and ν represent four-momenta.

$$\begin{aligned}\overline{\sum}|\mathcal{M}^{(t)}|^2 \quad &= \quad \left[\overline{\sum}|\mathcal{M}_0|^2 \times |D_0|^2 + \overline{\sum}|\mathcal{M}_L|^2 \times |D_L|^2\right] \\ &\quad \times \frac{\pi}{M_W\Gamma_W}\,\delta(w^2 - M_W^2).\end{aligned}$$

Here M_0, M_L are given above with $y = 0$ and D_0, D_L are the helicity amplitudes for the decay of a longitudinal and left-handed W boson respectively:

$$
\begin{aligned}
|D_0|^2 &= \frac{G_F M_W^4}{\sqrt{2}} \frac{1}{2} \sin^2 \theta_e^* \\
|D_L|^2 &= \frac{G_F M_W^4}{\sqrt{2}} \frac{1}{4} (1 - \cos \theta_e^*)^2 .
\end{aligned}
$$

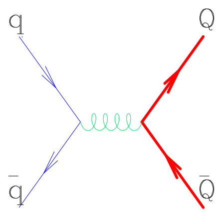


Heavy quark production, leading order

The leading-order processes for the production of a heavy quark Q of mass m in hadron-hadron collisions

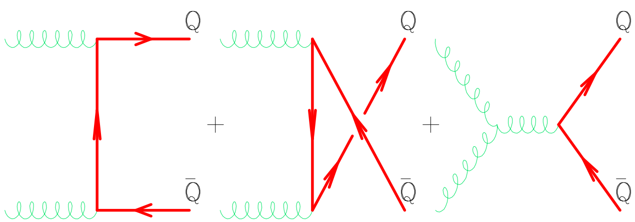
$$
\begin{array}{ll}
(a) & q(p_1) + \overline{q}(p_2) \to Q(p_3) + \overline{Q}(p_4) \\
(b) & g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4)
\end{array}
$$

where the four-momenta of the partons are given in brackets.



(a)

Process	$\overline{\sum}\lvert\mathcal{M}\rvert^2/g^4$
$q\,\overline{q} \to Q\,\overline{Q}$	$\frac{4}{9}\left(\tau_1^2+\tau_2^2+\frac{\rho}{2}\right)$
$g\,g \to Q\,\overline{Q}$	$\left(\frac{1}{6\tau_1\tau_2}-\frac{3}{8}\right)\left(\tau_1^2+\tau_2^2+\rho-\frac{\rho^2}{4\tau_1\tau_2}\right)$



(b)

$\overline{\sum}$ indicates averaged (summed) over initial (final) colours and spins

We have introduced the following notation for the ratios of scalar products:

$$\tau_1 = \frac{2p_1.p_3}{\hat{s}}, \quad \tau_2 = \frac{2p_2.p_3}{\hat{s}}, \quad \rho = \frac{4m^2}{\hat{s}}, \quad \hat{s} = (p_1+p_2)^2.$$

- The short-distance cross section is obtained from the invariant matrix element in the usual way:

$$d\hat{\sigma}_{ij} = \frac{1}{2\hat{s}} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4) \overline{\sum} |\mathcal{M}_{ij}|^2.$$

  The first factor is the flux factor for massless incoming particles. The other terms come from the phase space for $2 \to 2$ scattering.

- In terms of the rapidity $y = \frac{1}{2}\ln((E+p_z)/(E-p_z))$ and transverse momentum, p_T, the relativistically invariant phase space volume element of the final-state heavy quarks is

$$\frac{d^3p}{E} = dy\, d^2p_T \ .$$

The result for the invariant cross section may be written as

$$\frac{d\sigma}{dy_3 dy_4 d^2 p_T} = \frac{1}{16\pi^2 \hat{s}^2} \sum_{ij} x_1 f_i(x_1, \mu^2)\; x_2 f_j(x_2, \mu^2) \overline{\sum} |\mathcal{M}_{ij}|^2.$$

x_1 and x_2 are fixed if we know the transverse momenta and rapidity of the outgoing heavy quarks. In the centre-of-mass system of the incoming hadrons we may write

$$\begin{aligned}
p_1 &= \frac{1}{2}\sqrt{s}(x_1, 0, 0, x_1) \\
p_2 &= \frac{1}{2}\sqrt{s}(x_2, 0, 0, -x_2) \\
p_3 &= (m_T \cosh y_3, p_T, 0, m_T \sinh y_3) \\
p_4 &= (m_T \cosh y_4, -p_T, 0, m_T \sinh y_4).
\end{aligned}$$

Applying energy and momentum conservation, we obtain

$$x_1 = \frac{m_T}{\sqrt{s}}\left(e^{y_3} + e^{y_4}\right), x_2 = \frac{m_T}{\sqrt{s}}\left(e^{-y_3} + e^{-y_4}\right), \hat{s} = 2m_T^2(1 + \cosh \Delta y).$$

The quantity $m_T = \sqrt{(m^2 + p_T^2)}$ is the transverse mass of the heavy quarks and $\Delta y = y_3 - y_4$ is the rapidity difference between them.

In these variables the leading order cross section is

$$\frac{d\sigma}{dy_3 dy_4 d^2 p_T} = \frac{1}{64\pi^2 m_T^4 (1+\cosh(\Delta y))^2} \sum_{ij} x_1 f_i(x_1,\mu^2)\, x_2 f_j(x_2,\mu^2)\, \overline{\sum} |\mathcal{M}_{ij}|^2 .$$

Expressed in terms of m, m_T and Δy, the matrix elements for the two processes are

$$\overline{\sum} |\mathcal{M}_{q\bar{q}}|^2 = \frac{4g^4}{9} \Big(\frac{1}{1+\cosh(\Delta y)} \Big) \Big(\cosh(\Delta y) + \frac{m^2}{m_T^2} \Big) ,$$

$$\overline{\sum} |\mathcal{M}_{gg}|^2 = \frac{g^4}{24} \Big(\frac{8\cosh(\Delta y) - 1}{1+\cosh(\Delta y)} \Big) \Big(\cosh(\Delta y) + 2\frac{m^2}{m_T^2} - 2\frac{m^4}{m_T^4} \Big).$$

- As the rapidity separation Δy between the two heavy quarks becomes large

$$\overline{\sum} |\mathcal{M}_{q\bar{q}}|^2 \sim \text{ constant,} \quad \overline{\sum} |\mathcal{M}_{gg}|^2 \sim \exp \Delta y \; .$$

- The cross section is damped at large Δy and heavy quarks produced by $q\bar{q}$ annihilation are more closely correlated in rapidity those produced by gg fusion.

Applicability of perturbation theory?

- Consider the propagators in the diagrams.

$$
\begin{aligned}
(p_1+p_2)^2 = &\quad 2p_1.p_2 &= 2m_T^2\left(1+\cosh\Delta y\right)\,,\\
(p_1-p_3)^2-m^2 = &\quad -2p_1.p_3 &= -m_T^2\left(1+e^{-\Delta y}\right)\,,\\
(p_2-p_3)^2-m^2 = &\quad -2p_2.p_3 &= -m_T^2\left(1+e^{\Delta y}\right)\,.
\end{aligned}
$$

  Note that the propagators are all off-shell by a quantity of least of order m^2.
- Thus for a sufficiently heavy quark we expect the methods of perturbation theory to be applicable. It is the mass m (which by supposition is very much larger than the scale of the strong interactions Λ) which provides the large scale in heavy quark production. We expect corrections of order Λ/m
- This does not address the issue of whether the charm or bottom mass is large enough to be adequately described by perturbation theory.

Heavy quark production in $O(\alpha_S^3)$

In NLO heavy quark production m is the heavy quark mass.

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 \, \hat{\sigma}_{ij}(x_1 x_2 S, m^2, \mu^2) F_i(x_1, \mu^2) F_j(x_2, \mu^2)$$

$$\hat{\sigma}_{i,j}(\hat{s}, m^2, \mu^2) = \sigma_0 c_{ij}(\hat{\rho}, \mu^2)$$

where $\hat{\rho} = 4m^2/\hat{s}, \bar{\mu}^2 = \mu^2/m^2, \sigma_0 = \alpha_S^2(\mu^2)/m^2$ and $\hat{s}$ in the parton total c-of-m energy squared. The coupling satisfies

$$\frac{d\alpha_S}{d\ln\mu^2} = -b_0 \frac{\alpha_S^2}{2\pi} + O(\alpha_S^3), \;\; b_0 = \frac{11N - 2n_f}{6}$$

$$c_{ij}\left(\rho, \frac{\mu^2}{m^2}\right) = c_{ij}^{(0)}(\rho) + 4\pi\alpha_S(\mu^2)\left[c_{ij}^{(1)}(\rho) + \overline{c}_{ij}^{(1)}(\rho)\ln(\frac{\mu^2}{m^2})\right] + O(\alpha_S^2)$$

The lowest-order functions $c_{ij}^{(0)}$ are obtained by integrating the lowest order matrix elements

$$c_{q\overline{q}}^{(0)}(\rho) = \frac{\pi\beta\rho}{27}\left[(2+\rho)\right] ,$$

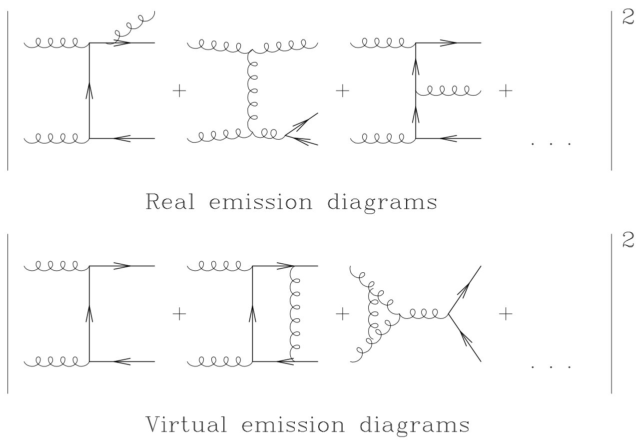
$$c_{gg}^{(0)}(\rho) = \frac{\pi\beta\rho}{192}\left[\frac{1}{\beta}\left[\rho^2+16\rho+16\right]\ln\left(\frac{1+\beta}{1-\beta}\right)-28-31\rho\right] ,$$

$$c_{gq}^{(0)}(\rho) = c_{g\overline{q}}^{(0)}(\rho) = 0 ,$$

and $\beta = \sqrt{1-\rho}$.

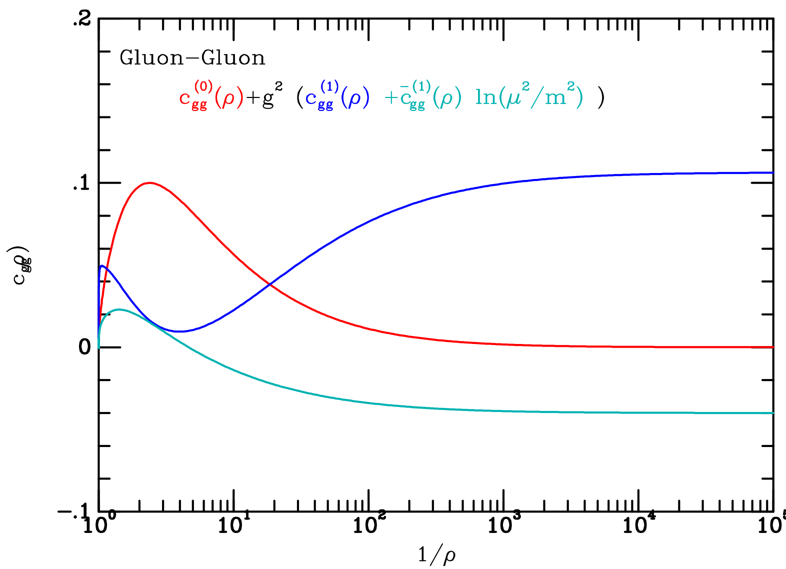
- The functions $c_{ij}^{(0)}$ vanish both at threshold ($\beta \to 0$) and at high energy ($\rho \to 0$).
- Note that the quark-gluon process is zero in lowest order, but is present in higher orders.

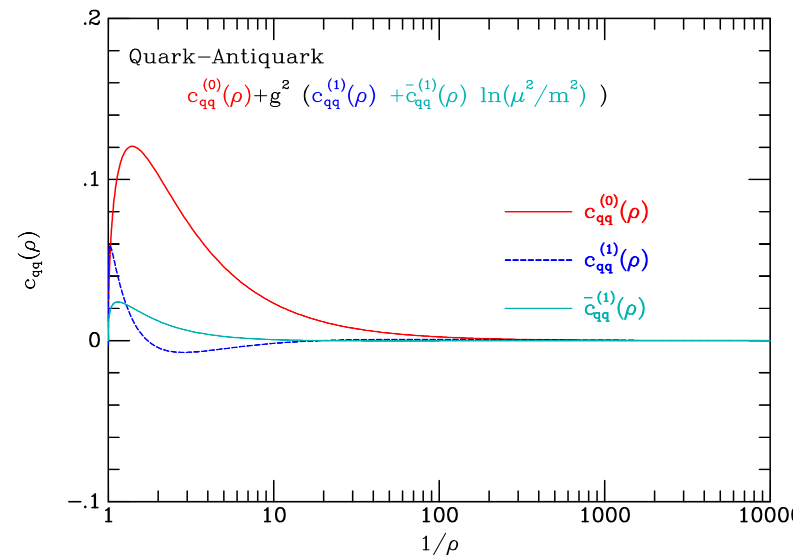
- The functions $c_{ij}^{(1)}$ are also known

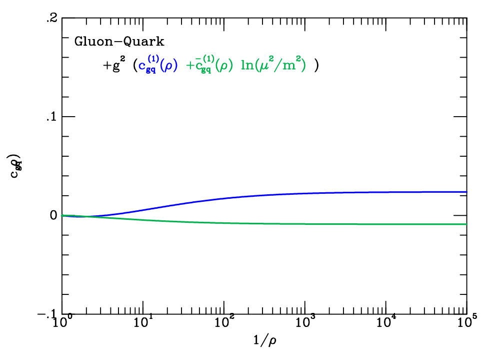


- Examples of higher-order corrections to heavy quark production.
- In order to calculate the c_{ij} in perturbation theory we must perform both renormalization and factorization of mass singularities. The subtractions required for renormalization and factorization are done at mass scale μ.

Higher order results, $c_{ij}^{(1)}$







μ dependence

μ is an unphysical parameter. The physical predictions should be invariant under changes of μ at the appropriate order in perturbation theory. If we have performed a calculation to $O(\alpha_S^3)$, variations of the scale μ will lead to corrections of $O(\alpha_S^4)$,

$$\mu^2 \frac{d}{d\mu^2}\sigma = O(\alpha_S^4).$$

- The term $\overline{c}^{(1)}$, which controls the μ dependence of the higher-order perturbative contributions, is fixed in terms of the lower-order result $c^{(0)}$:

$$\begin{aligned}\overline{c}_{ij}^{(1)}(\rho) &= \frac{1}{8\pi^2}\Bigg[4\pi b c_{ij}^{(0)}(\rho) - \int_\rho^1 dz_1 \sum_k c_{kj}^{(0)}(\frac{\rho}{z_1}) P_{ki}^{(0)}(z_1) \\ &- \int_\rho^1 dz_2 \sum_k c_{ik}^{(0)}(\frac{\rho}{z_2}) P_{kj}^{(0)}(z_2)\Bigg].\end{aligned}$$

In obtaining this result we have used the renormalization group equation for the running coupling

$$\mu^2 \frac{d}{d\mu^2} \alpha_S(\mu^2) = -b\alpha_S^2 + \dots$$

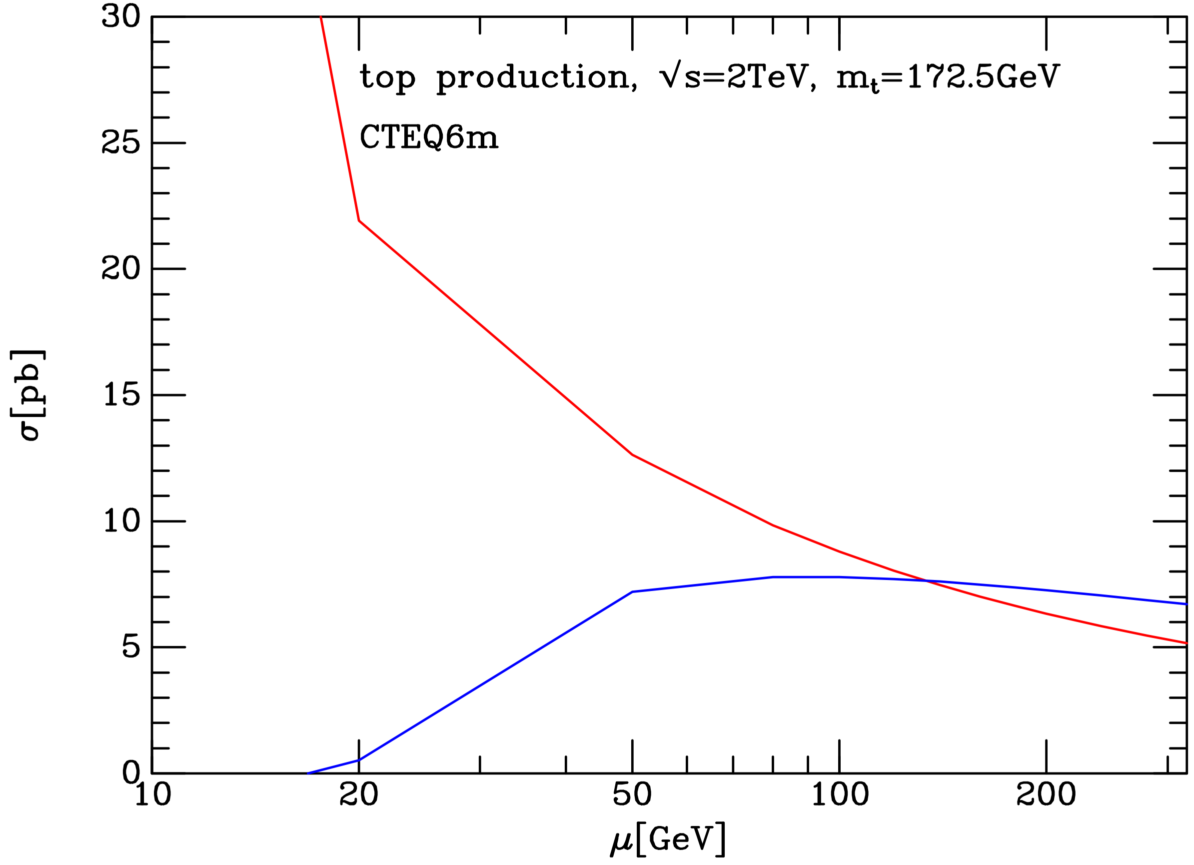
and the lowest-order form of the GLAP equation

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \frac{\alpha_S(\mu^2)}{2\pi} \sum_k \int_x^1 \frac{dz}{z} P_{ik}^{(0)}(z) f_k(\frac{x}{z}, \mu^2) + \dots .$$

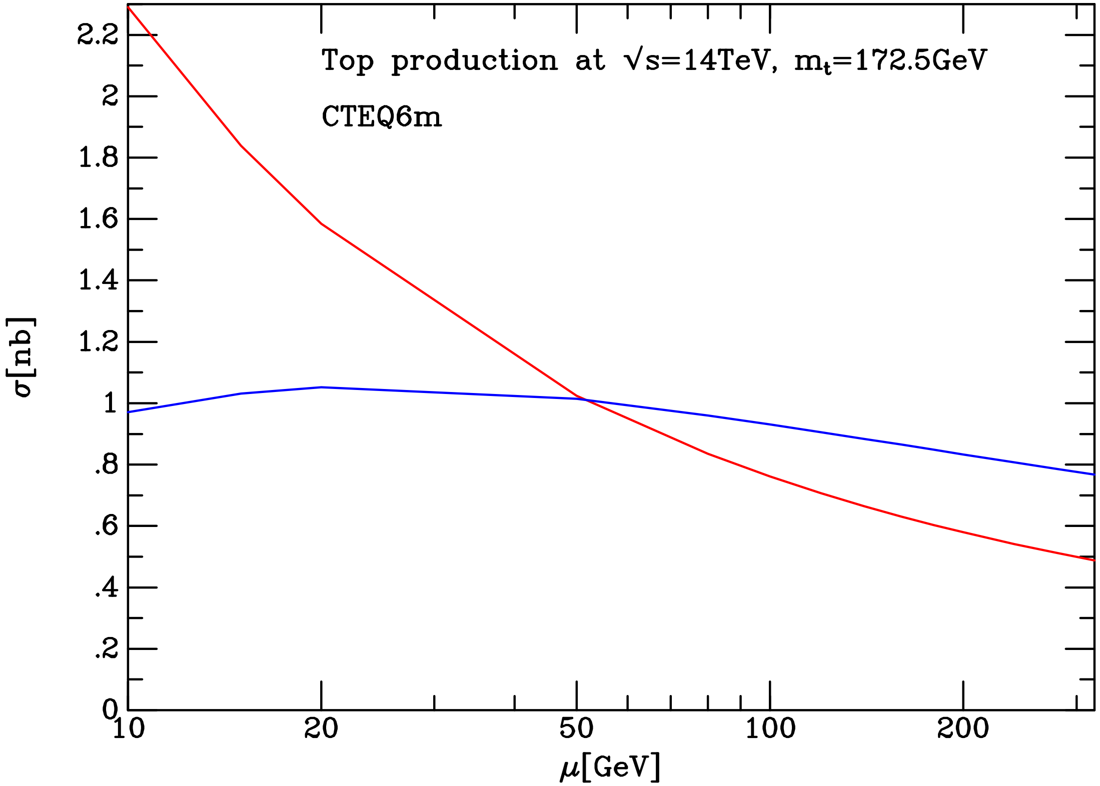
This illustrates an important point which is a general feature of renormalization group improved perturbation series in QCD. The coefficient of the perturbative correction depends on the choice made for the scale μ, but the scale dependence changes the result in such a way that the physical result is independent of that choice. Thus the scale dependence is formally small because it is of higher order in α_S. This does not assure us that the scale dependence is actually *numerically* small for all series. A pronounced dependence on the scale μ is a signal of an untrustworthy perturbation series.

Scale dependence in top production

- Inclusion of the higher order terms leads to a stabilization of the top cross section.



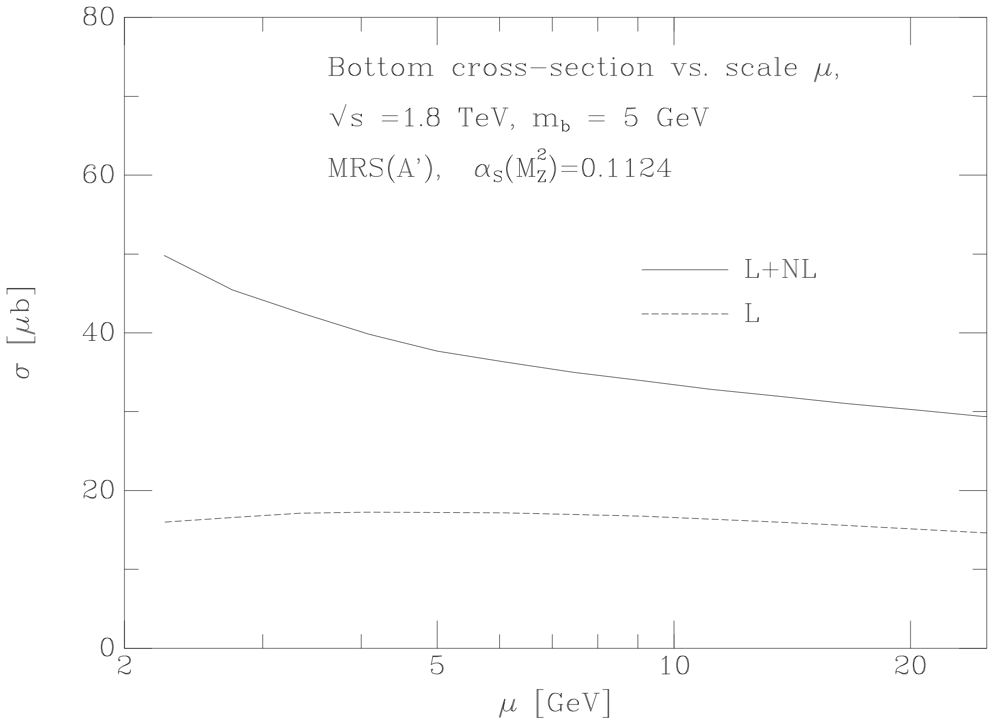
Top production at LHC



- At LHC top cross section is more than 100 times bigger than at Tevatron.

Scale dependence in bottom production

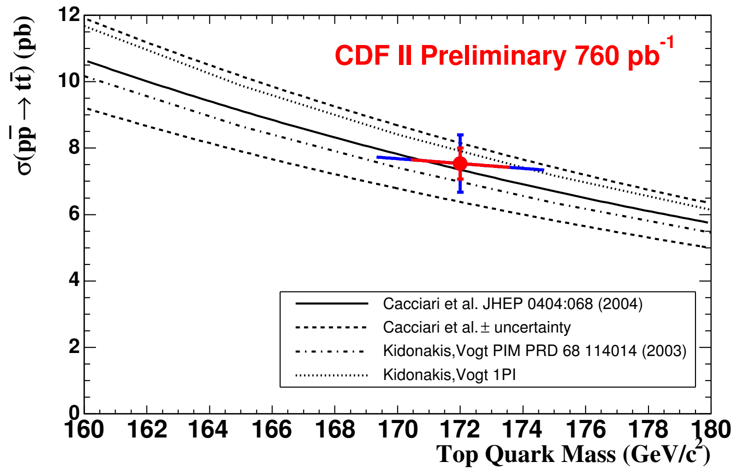
- The perturbation series for bottom quark production is not well behaved.
- The lowest order cross section is almost μ independent because of an accidental cancellation between the fall-off of α_S and the increase of the gluon distributions with increasing μ.



Top production

- All the information on the top quark is still rather limited and crude
- Within errors agreement between three generation theory and experiment

	Experiment	Theory
m_t	172.5 ± 2.3 GeV	$178.9 + 12.0 - 9.0$ GeV
$BR(t \to Wb)/BR(t \to Wq)$	$1.11^{+0.21}_{-0.26}$	≈ 1
$BR(t \to W_0 b)$	$0.74^{+0.22}_{-0.34}$	≈ 0.7
$BR(t \to W_+ b)$	< 0.27 (95%cl)	≈ 0



Recap

- NLO formulation of QCD processes gives better information about normalization, and less dependence on unphysical scales.
- Matching with Monte Carlo can be implemented (Giele)
- Simple spectator model gives a poor description of hadronic charm decays and a good description of bottom decay.
- Top decay is a semi-weak process
- Heavy quark production is calculable in perturbation theory inasmuch as m_Q is greater than Λ QCD.
- Bottom quark production gives a troubled perturbation series, but with care on the fragmentation, a fair agreement can be found.
- Top quark production at the Tevatron is in good agreement with theory.
- The top quark is produced at LHC at a large rate.